



INTERNATIONAL  
BACCALAUREATE  
ORGANIZATION

**GROUP 5 MATHEMATICS**  
**FORMULAE AND STATISTICAL TABLES**

To be used in the teaching and examination of:  
Mathematics HL  
Mathematical Methods SL  
Mathematical Studies SL  
Further Mathematics SL

*Third Edition: February 2001*  
*Valid for Examination Sessions from May 2001*

*Group 5 Mathematics  
Formulae and Statistical Tables  
First published: August 1998  
Second edition: April 1999  
Third edition: February 2001*

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International Baccalaureate Organization  
Route des Morillons 15  
1218 Grand-Saconnex  
Geneva, SWITZERLAND

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## 1 Plane and Solid Figures

Area of a parallelogram:	$A = (b \times h)$ , where $b$ is the base, $h$ is the height
Area of a triangle:	$A = \frac{1}{2}(b \times h)$ , where $b$ is the base, $h$ is the height
Area of a trapezium:	$A = \frac{1}{2}(a + b)h$ , where $a$ and $b$ are the parallel sides, $h$ is the height
Area of a circle:	$A = \pi r^2$ , where $r$ is the radius
Circumference of a circle:	$C = 2\pi r$ , where $r$ is the radius
Volume of a pyramid:	$V = \frac{1}{3}(\text{area of base} \times \text{vertical height})$
Volume of a cuboid:	$V = l \times w \times h$ , where $l$ is the length, $w$ is the width, $h$ is the height
Volume of a cylinder:	$V = \pi r^2 h$ , where $r$ is the radius, $h$ is the height
Area of the curved surface of a cylinder:	$A = 2\pi r h$ , where $r$ is the radius, $h$ is the height
Volume of a sphere:	$V = \frac{4}{3}\pi r^3$ , where $r$ is the radius
Volume of a cone:	$V = \frac{1}{3}\pi r^2 h$ , where $r$ is the radius, $h$ is the height

## 2 Finite Sequences

The $n$ th term of an arithmetic sequence:	$u_n = u_1 + (n - 1)d$
The sum of $n$ terms of an arithmetic sequence:	$S_n = \frac{n}{2}(2u_1 + (n - 1)d) = \frac{n}{2}(u_1 + u_n)$
The $n$ th term of a geometric sequence:	$u_n = u_1 r^{n-1}$
The sum of $n$ terms of a geometric sequence:	$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$ , $r \neq 1$

## 3 Trigonometry

Sine rule:	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
Cosine rule:	$a^2 = b^2 + c^2 - 2bc \cos A$ ; $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
Area of a triangle:	$A = \frac{1}{2}ab \sin C$ , where $a$ and $b$ are adjacent sides, $C$ is the included angle

#### 4 Geometry

Distance between two points  $(x_1, y_1)$  and  $(x_2, y_2)$ :  $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Coordinates of the midpoint of a line segment with endpoints  $(x_1, y_1)$  and  $(x_2, y_2)$ :  $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Magnitude of a vector:  $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$ , where  $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

#### 5 Financial Mathematics

Simple interest:  $I = \frac{Crn}{100}$ , where  $C$  is the capital,  $r\%$  is the interest rate,  $n$  is the number of time periods,  $I$  is the interest

Compound interest:  $I = C\left(1 + \frac{r}{100}\right)^n - C$ , where  $C$  is the capital,  $r\%$  is the interest rate,  $n$  is the number of time periods,  $I$  is the interest

#### 6 Matrices ( $2 \times 2$ )

Determinant:  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = ad - bc$

Transpose:  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

#### 7 Probability

Probability of an event  $A$ :  $P(A) = \frac{n(A)}{n(U)}$

Complementary events:  $P(A') = 1 - P(A)$

Combined events:  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Mutually exclusive events:  $P(A \cup B) = P(A) + P(B)$

Independent events:  $P(A \cap B) = P(A)P(B)$

Conditional probability:  $P(A|B) = \frac{P(A \cap B)}{P(B)}$

## 8 Statistics

Population mean: 
$$\mu = \frac{\sum_{i=1}^k f_i x_i}{n}, \text{ where } n = \sum_{i=1}^k f_i$$

Population standard deviation: 
$$\sigma = \sqrt{\frac{\sum_{i=1}^k f_i (x_i - \mu)^2}{n}}, \text{ where } n = \sum_{i=1}^k f_i$$

Sample mean: 
$$\bar{x} = \frac{\sum_{i=1}^k f_i x_i}{n}, \text{ where } n = \sum_{i=1}^k f_i$$

Standard deviation of the sample: 
$$s_n = \sqrt{\frac{\sum_{i=1}^k f_i (x_i - \bar{x})^2}{n}}, \text{ where } n = \sum_{i=1}^k f_i$$

Standardized normal variable: 
$$z = \frac{x - \mu}{\sigma}$$

Covariance: 
$$s_{xy} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{n}$$

Product moment correlation coefficient: 
$$r = \frac{s_{xy}}{s_x s_y}, \text{ where } s_x = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}}, \quad s_y = \sqrt{\frac{\sum_{i=1}^n (y_i - \bar{y})^2}{n}}$$

Regression line for y on x: 
$$y - \bar{y} = \frac{s_{xy}}{s_x^2} (x - \bar{x})$$

The  $\chi^2$  test statistic: 
$$\chi_{calc}^2 = \sum \frac{(f_e - f_o)^2}{f_e}, \text{ where } f_e \text{ are the expected frequencies,}$$
  

$$f_o \text{ are the observed frequencies}$$

## 9 Differential Calculus

Derivative of  $f(x)$  :  $y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) = \lim_{h \rightarrow 0} \left( \frac{f(x+h) - f(x)}{h} \right)$

Derivative of  $ax^n$  :  $f(x) = ax^n \Rightarrow f'(x) = nax^{n-1}$

Derivative of a polynomial:  $f(x) = ax^n + bx^{n-1} + \dots \Rightarrow f'(x) = nax^{n-1} + (n-1)bx^{n-2} + \dots$

Anti-differentiation:  $\frac{dy}{dx} = x^n \Rightarrow y = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

## 10 Infinite Sequences

The sum of an infinite geometric sequence:  $S = \frac{u_1}{1-r}$ ,  $|r| < 1$

## 11 Algebra

Solution of a quadratic equation:  $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ,  $a \neq 0$

Exponents and logarithms:

$$a^x = b \Leftrightarrow x = \log_a b$$

$$a^x = e^{x \ln a}$$

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_b a = \frac{(\log_c a)}{(\log_c b)}$$

Binomial theorem:  $(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \dots + \binom{n}{r} a^{n-r} b^r + \dots + b^n$

## 12 Trigonometry

Length of an arc:  $l = \theta r$ , where  $\theta$  is the angle measured in radians,  $r$  is the radius

Area of a sector:  $A = \frac{1}{2} \theta r^2$ , where  $\theta$  is the angle measured in radians,  $r$  is the radius

Identities:

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$$

## 13 Vectors

Scalar product:  $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos \theta = v_1 w_1 + v_2 w_2$ , where  $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ ,  $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$\cos \theta = \frac{v_1 w_1 + v_2 w_2}{|\mathbf{v}| |\mathbf{w}|}$$

$$|\mathbf{v}| \cos \theta = \mathbf{v} \cdot \left( \frac{\mathbf{w}}{|\mathbf{w}|} \right)$$

Vector equation of a line:  $\mathbf{r} = \mathbf{p} + t\mathbf{d}$

### 14 Matrices ( $2 \times 2$ )

Inverse: 
$$\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \mathbf{P}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Transformation matrix representing a rotation through  $\theta$  about the origin: 
$$\mathbf{R} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Transformation matrix representing a reflection in  $y = x \tan \theta$ : 
$$\mathbf{M} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

### 15 Statistics

Standard error of the mean: 
$$SE = \frac{\sigma}{\sqrt{n}}$$

Test statistic for the mean of a normal population: 
$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

### 16 Differentiation

Derivative of  $\sin x$ :  $f(x) = \sin x \Rightarrow f'(x) = \cos x$

Derivative of  $\cos x$ :  $f(x) = \cos x \Rightarrow f'(x) = -\sin x$

Derivative of  $e^x$ :  $f(x) = e^x \Rightarrow f'(x) = e^x$

Derivative of  $\ln x$ :  $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

Derivative of  $a^x$ :  $f(x) = a^x \Rightarrow f'(x) = a^x (\ln a)$

Derivative of  $\log_a x$ :  $f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x \ln a}$

Derivative of  $\tan x$ :  $f(x) = \tan x \Rightarrow f'(x) = \frac{1}{\cos^2 x}$

Chain rule:  $y = g(u)$ , where  $u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Product rule:  $y = uv \Rightarrow \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

Quotient rule:  $y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

### 17 Integration

Standard integrals:  $\int \frac{1}{x} dx = \ln x + C$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int e^x dx = e^x + C$$

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + C, \quad n \neq -1$$

### 18 Iteration

Newton–Raphson method:  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

### 19 Approximate Integration

Trapezium rule:  $\int_a^b f(x) dx \approx \frac{h}{2} [y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n]$

where  $h = \frac{b-a}{n}$ ;  $y_i = f(a + ih)$ ,  $i = 0, 1, 2, \dots, n$

## 20 Combinations

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

## 21 Series

The sum of the first  $n$  integers:  $\sum_{i=1}^n i = \frac{n}{2}(n+1)$

The sum of the squares of the first  $n$  integers:  $\sum_{i=1}^n i^2 = \frac{n}{6}(n+1)(2n+1)$

The sum of the cubes of the first  $n$  integers:  $\sum_{i=1}^n i^3 = \frac{n^2}{4}(n+1)^2$

## 22 Complex Numbers

$$z = a + ib = r(\cos \theta + i \sin \theta)$$

De Moivre's theorem:  $[r(\cos \theta + i \sin \theta)]^n = r^n (\cos n\theta + i \sin n\theta)$

## 23 Trigonometry

Identities:  $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

$$\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$$

Compound formula:  $a \cos x \pm b \sin x = R \cos(x \mp \alpha)$ , where  $R = \sqrt{a^2 + b^2}$ ,  $\tan \alpha = \frac{b}{a}$

## 24 Vector Geometry

Magnitude of a vector:  $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ , where  $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

Scalar product:  $\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$ , where  $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ ,  $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$$\cos \theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{|\mathbf{v}||\mathbf{w}|}$$

Vector product:  $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}|\sin \theta$$

Area of a triangle:  $A = \frac{1}{2}|\mathbf{v} \times \mathbf{w}|$

Vector equation of a line:  $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

Vector equation of a plane:  $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$

Equation of a plane  
(using the normal vector):  $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

Cartesian equation of a line:  $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$

Cartesian equation of a plane:  $ax + by + cz + d = 0$

## 25 Matrices (3×3)

Determinant:  $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \Rightarrow \det A = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

## 26 Differentiation

Derivative of  $\sec x$  :  $f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$

Derivative of  $\csc x$  :  $f(x) = \csc x \Rightarrow f'(x) = -\csc x \cot x$

Derivative of  $\cot x$  :  $f(x) = \cot x \Rightarrow f'(x) = -\csc^2 x$

Derivative of  $\arcsin x$  :  $f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$

Derivative of  $\arccos x$  :  $f(x) = \arccos x \Rightarrow f'(x) = -\frac{1}{\sqrt{1-x^2}}$

Derivative of  $\arctan x$  :  $f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$

## 27 Integration

Integration by parts:  $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

Standard integrals:  $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C, \quad |x| < a$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

## 28 Approximate Integration

Trapezium rule  
(including error term):  $\int_a^b f(x) dx = \frac{h}{2} \left[ y_0 + 2 \sum_{i=1}^{n-1} y_i + y_n \right] - \frac{(b-a)h^2}{12} f''(c)$

where  $h = \frac{b-a}{n}$ ;  $y_i = f(a+ih)$ ,  $i = 0, 1, 2, \dots, n$ ;  $c \in ]a, b[$

Simpson's rule,  
for  $n$  even  
(including error term):  $\int_a^b f(x) dx = \frac{h}{3} [y_0 + 4y_1 + 2y_2 + \dots + 2y_{n-2} + 4y_{n-1} + y_n] - \frac{(b-a)h^4}{180} f^{(4)}(c)$

where  $h = \frac{b-a}{n}$ ;  $y_i = f(a+ih)$ ,  $i = 0, 1, 2, \dots, n$ ;  $c \in ]a, b[$

## 29 Probability

Expected value of a discrete random variable  $X$ :  $E(X) = \mu = \sum x P(X = x)$

Expected value of a continuous random variable  $X$ :  $E(X) = \mu = \int x f(x) dx$

Variance:  $\text{Var}(X) = E(X - \mu)^2 = E(X^2) - [E(X)]^2$

Poisson distribution:  $X \sim P(\mu) \Rightarrow P(X = r) = \frac{\mu^r e^{-\mu}}{r!}, r = 0, 1, 2, \dots$

Binomial distribution:  $X \sim B(n, p) \Rightarrow P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}, r = 0, 1, \dots, n$

Bayes' Theorem:  $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

## 30 Statistics

Linear combinations of two random variables  $X_1, X_2$ :  $E(a_1 X_1 \pm a_2 X_2) = a_1 E(X_1) \pm a_2 E(X_2)$

Linear combinations of two independent random variables  $X_1, X_2$ :  $\text{Var}(a_1 X_1 \pm a_2 X_2) = a_1^2 \text{Var}(X_1) + a_2^2 \text{Var}(X_2)$

Unbiased estimate of the population variance:  $s_{n-1}^2 = \frac{n}{n-1} s_n^2 = \frac{\sum_{i=1}^k f_i (x_i - \bar{x})^2}{n-1}$ ,

$$\text{where } n = \sum_{i=1}^k f_i$$

Pooled estimate of the population mean for two samples of size  $n$  and  $m$ :  $\bar{x} = \frac{n\bar{x}_n + m\bar{x}_m}{n+m}$

Pooled estimate of the population variance for two samples of size  $n$  and  $m$ :  $s_{n+m-2}^2 = \frac{ns_n^2 + ms_m^2}{n+m-2} = \frac{(n-1)s_{n-1}^2 + (m-1)s_{m-1}^2}{n+m-2}$

Test statistic for the mean of a normal population of unknown variance:  $t = \frac{\bar{x} - \mu}{s_n / \sqrt{n-1}} = \frac{\bar{x} - \mu}{s_{n-1} / \sqrt{n}}$

### 31 Series and Approximation

Maclaurin series:  $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$

Taylor series:  $f(a+x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \dots$

Taylor approximations (including error term):  $f(a+x) = f(a) + x f'(a) + \dots + \frac{x^n}{n!} f^{(n)}(a) + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c)$   
 where  $c$  is between  $a$  and  $a+x$ , (excluding endpoints).

### 32 Set Theory

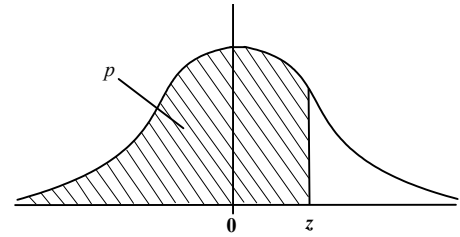
De Morgan's Laws:  $(A \cup B)' = A' \cap B'$   
 $(A \cap B)' = A' \cup B'$

### 33 Graph Theory

Euler's relation:  $v - e + f = 2$ , where  $v$  is the number of vertices,  $e$  is the number of edges,  
 $f$  is the number of faces

**TABLE I: AREA UNDER THE STANDARD NORMAL CURVE**

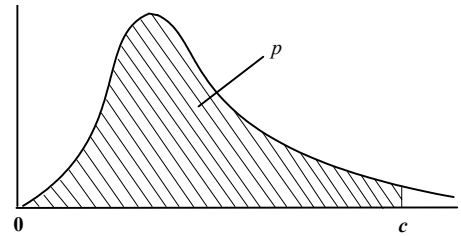
$$p = P(Z \leq z)$$



<i>z</i>	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8079	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9892	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9983	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9988	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998

**TABLE 2: CRITICAL VALUES OF THE  $\chi^2$  DISTRIBUTION**

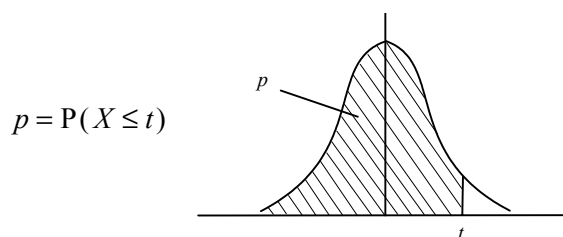
$$p = P(X \leq c)$$



$\nu$	0.005	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99	0.995
1	0.00004	0.0002	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

$\nu$  = number of degrees of freedom

**TABLE 3: CRITICAL VALUES OF THE STUDENT'S  $t$ -DISTRIBUTION**



$p$	0.9	0.95	0.975	0.99	0.995	0.9995
$\nu=1$	3.078	6.314	12.706	31.821	63.657	636.619
2	1.886	2.920	4.303	6.965	9.925	31.599
3	1.638	2.353	3.182	4.541	5.841	12.924
4	1.533	2.132	2.776	3.747	4.604	8.610
5	1.476	2.015	2.571	3.365	4.032	6.869
6	1.440	1.943	2.447	3.143	3.707	5.959
7	1.415	1.895	2.365	2.998	3.499	5.408
8	1.397	1.860	2.306	2.896	3.355	5.041
9	1.383	1.833	2.262	2.821	3.250	4.781
10	1.372	1.812	2.228	2.764	3.169	4.587
11	1.363	1.796	2.201	2.718	3.106	4.437
12	1.356	1.782	2.179	2.681	3.055	4.318
13	1.350	1.771	2.160	2.650	3.012	4.221
14	1.345	1.761	2.145	2.624	2.977	4.140
15	1.341	1.753	2.131	2.602	2.947	4.073
16	1.337	1.746	2.120	2.583	2.921	4.015
17	1.333	1.740	2.110	2.567	2.898	3.965
18	1.330	1.734	2.101	2.552	2.878	3.922
19	1.328	1.729	2.093	2.539	2.861	3.883
20	1.325	1.725	2.086	2.528	2.845	3.850
21	1.323	1.721	2.080	2.518	2.831	3.819
22	1.321	1.717	2.074	2.508	2.819	3.792
23	1.319	1.714	2.069	2.500	2.807	3.768
24	1.318	1.711	2.064	2.492	2.797	3.745
25	1.316	1.708	2.060	2.485	2.787	3.725
26	1.315	1.706	2.056	2.479	2.779	3.707
27	1.314	1.703	2.052	2.473	2.771	3.690
28	1.313	1.701	2.048	2.467	2.763	3.674
29	1.311	1.699	2.045	2.462	2.756	3.659
30	1.310	1.697	2.042	2.457	2.750	3.646
40	1.303	1.684	2.021	2.423	2.704	3.551
60	1.296	1.671	2.000	2.390	2.660	3.460
120	1.289	1.658	1.980	2.358	2.617	3.373
***	1.282	1.645	1.960	2.326	2.576	3.291

$\nu$  = number of degrees of freedom